On the Coalgebra Description of OCHA

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Abstract

OCHA is the homotopy algebra of open-closed strings. It can be defined as a sequence of multilinear operations on a pair of DG spaces satisfying certain relations which include the L_∞ relations in one space and the A_∞ relations in the other. In this paper we show that the OCHA structure is intrinsic to the tensor product of the symmetric and tensor coalgebras. We also show how an OCHA can be obtained from A_∞-extesions and define the *universal enveloping* A_∞-algebra of an OCHA as an A_∞-extension of the universal enveloping of its L_∞ part by its A_∞ part.

Contents

1 Introduction

Inspired by Zwiebach's classical open-closed string field theory, Kajiura and Stasheff introduced **O**pen-**C**losed **H**omotopy **A**lgebras [6]. OCHAs were presented in three equivalent ways: as a sequence of *multilinear operations* satisfying certain conditions; as an *algebra over a DG Operad* and as a *coderivation differential* on a certain coalgebra.

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Let $(\mathcal{H}_c, \mathcal{H}_o)$ be a pair of DG spaces. According to the *multilinear operations* description, an OCHA structure on the pair $(\mathcal{H}_c, \mathcal{H}_o)$ consists of two sequences of multilinear maps: $l_n : \mathcal{H}_c^{\wedge n} \to \mathcal{H}_c$, $n \geqslant 1$ and $n_{p,q} : \mathcal{H}_c^{\wedge p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_o$, $p+q \geqslant 1$, satisfying certain compatbility conditions. The compatibility conditions say that the maps $\{l_n\}$ define an L_∞ structure on $\mathcal{H}_c$, the maps $\{n_{0,q}\}$ define an A_∞ structure on $\mathcal{H}_o$ and the maps $\{n_{p,q}\}$ for $p, q \geqslant 1$ provide the strucutre of an A_∞-algebra over an L_∞-algebra on $\mathcal{H}_o$ (i.e., the strongly homotopy version of the action by derivations of a Lie algebra on an associative algebra, see [6]). It remais to understand mathematically the terms of the form $n_{p,0} : \mathcal{H}_c^{\wedge p} \to \mathcal{H}_o$. Providing one possible *mathematical* interpratation for those maps is one of the concerns of the present paper. On the other hand, their physical meaning is well recognized. According to [6], the maps $n_{p,0}$ originated from the operations of *opening closed strings into open ones.*

The *operadic description* consists of providing a differential graded operad whose algebras (or representations) are precisely those pairs of DG spaces endowed with the structure of an OCHA. Kajiura and Stasheff defined that operad using the language of trees in [7] and discussed the geometry behind that operad in [8]. That geometrical description was further studied in [3] and used to give the OCHA operad a description in terms of minimal resolutions involving the Swiss-Cheese operad. There we prove that the OCHA operad is the operad defined by the first row of the E^1 term of the spectral sequence of the compactified configuration space of points on the closed disc.

As for the *coderivation description* (or *coalgebra description* of OCHA), let us consider the multlinear maps: $l_n : \mathcal{H}_c^{\wedge n} \to \mathcal{H}_c$ and $n_{p,q} : \mathcal{H}_c^{\wedge p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_o$ satisfying the above mentioned compatbility conditions. Kajiura and Stasheff showed that, after lifting those maps as coderivations $\mathfrak{l}$ and $\mathfrak{n}$ on the coalgebra $\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o$, the compatbility condition is equivalent to the condition $(\mathfrak{l}+\mathfrak{n})^2 = 0$. That coalgebra description is the subject of the present paper. We will show that *any* coderivation on $\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o$ has the form $\mathfrak{l}+\mathfrak{n}$, thus showing that the OCHA structure is *intrinsic* to the tensor product of the symmetric and tensor coalgebras.

Main Results

We now describe the main results of this paper. The notation used will be explained at the end of this introduction. Let us assume a fixed field k of characteristic zero. Consider a vector space E with a splitting $E = A \oplus B$ in the category of vector spaces. Let $M = \{m_k : E^{\otimes k} \to E\}$ be a family of multilinear maps on E. We say that M satisfies the OCHA constraint with respect to the spliting $E = A \oplus B$ if it has only components of the form $M^A_{p,q} : A^{\otimes p} \otimes B^{\otimes p} \to A$ and $M^B_n : B^{\otimes n} \to B$. We say that a coderivation satisfies the constraint if it is obtained by lifting a family of maps that satisfy the constraint. More details are given in Section 3, where we prove the following result for $\mathcal{H}_c \oplus \mathcal{H}_o$.

Proposition. *Any coderivation on the coalgebra $\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o$ satisfies the OCHA constraint.*

This fact means that the OCHA constraint, whose geometrical and physical meaning is discussed in [3,8], is in fact *intrinsic* to the coalgebra $\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o$. As a consequence, we have the following theorem.

Theorem. *An OCHA structure on the pair of spaces $(\mathcal{H}_c, \mathcal{H}_o)$ is equivalent to a degree one coderivation differential $\mathcal{D} \in \mathrm{Coder}(\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o)$, $\mathcal{D}^2 = 0$.*

Section 4 involves a coalgebra map

$$\Xi : \Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \to T^c(\mathcal{H}_c \oplus \mathcal{H}_o)$$

given by symmetrization and shuffling. An A_∞-extension is a short exact sequence of A_∞-algebras $0 \to A \to E \to B \to 0$, where each map is a linear A_∞-morphism. The following theorem is proven in Section 4:

Theorem. *If $(E, \mathcal{D})$ is an A_∞-extension of B by A, then $\mathcal{D} \circ \Xi$ defines an OCHA structure on the pair (B, A). The L_∞-structure on B is the Lada-Markl symmetrization of the A_∞-structure on B.*

If the A_∞-extension $(E, \mathcal{D})$ splits, then the OCHA structure induced by the above theorem reduces to an A_∞-algebra over an L_∞-algebra. Such structures were introduced by Kajiura and Stasheff in [6] as the strong homotopy version of actions by derivations of Lie algebras on associative algebras. Structures containing pairs (L, A) where the Lie algebra L acts by derivations on the associative algebra A have appeared in different contexts in the literature, see [5] for an overview. If A is commutative, L is an A-module and L acts by derivations on A, then the pair (L, A) is called a *Lie-Rinehart Algebra*, see [4]. In the case where A is not necessarily commutative and L does not need to be an A-module, the pair is called a *Leibniz Pair* by Flato, Gerstenhaber and Voronov [2]. Thus, A_∞-algebras over L_∞-algebras are *Strong Homotopy Leibniz Pairs.*

The universal enveloping A_∞-algebra of an OCHA is introduced in Section 4.2. Its definition is completely analogous to the universal enveloping A_∞-algebra of an L_∞-algebra introduced by Lada and Markl in [10]. Given an OCHA $(\mathcal{H}_c, \mathcal{H}_o, \mathcal{D})$, its universal enveloping A_∞-algebra is denoted $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)$, while $\mathcal{U}_\infty(\mathcal{H}_c)$ denotes Lada-Markl A_∞-algebra. In section 4.2 we prove the following theorem:

Theorem. *The universal enveloping A_∞-algebra $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)$ of an OCHA $(\mathcal{H}_c, \mathcal{H}_o, \mathcal{D})$ is an A_∞-extension of $\mathcal{U}_\infty(\mathcal{H}_c)$ by $\langle \mathcal{H}_o \rangle$:*

$$0 \to \langle \mathcal{H}_o \rangle \to \mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o) \to \mathcal{U}_\infty(\mathcal{H}_c) \to 0$$

where $\langle \mathcal{H}_o \rangle$ denotes the A_∞-ideal generated by $\mathcal{H}_o$.

We close Section 4 by showing that the universal enveloping A_∞-algebra of an OCHA satisfies a universal property naturally described in terms of A_∞-extensions.

Notation and Conventions

Let us fix a field k of characteristic zero. By a vector space we will always mean a $\mathbb{Z}$-graded vector space. Let V be a vector space, we define a left action of the symmetric group S_n on $V^{\otimes n}$ in the following way: if $\tau \in S_2$ is a transposition, then the action is given by $x_1 \otimes x_2 \overset{\tau}{\mapsto} (-1)^{|x_1||x_2|} x_2 \otimes x_1$. Since any $\sigma \in S_n$ is a composition of transpositions, the sign of the action of σ on $V^{\otimes n}$ is well defined:

$$x_1 \otimes \cdots \otimes x_n \overset{\sigma}{\mapsto} (-1)^{\epsilon(\sigma)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

We will refer to $(-1)^{\epsilon(\sigma)}$ as the Koszul sign of the permutation. Let us define $(-1)^{\chi(\sigma)} = (-1)^{\sigma}(-1)^{\epsilon(\sigma)}$, where $(-1)^{\sigma}$ is the sign of the permutation. Given two homogeneous maps $f, g : V \to W$, we will follow the Koszul sign convention for the tensor product: $(f \otimes g)(v_1 \otimes v_2) = (-1)^{|g||v_1|}(f(v_1) \otimes g(v_2))$.

In general, a Strong Homotopy Algebra defined on some vector space V will be denoted as a pair $(V, \mathcal{D})$. The symbol $\mathcal{D}$ stands for the SH-structure. In this work, $\mathcal{D}$ can be thought of as a sequence of multilinear operations as well as a coderivation differential on some coalgebra. We observe that any vector space endowed with some SH structure has in particular a differential operator which is part of the SH structure.

2 Open Closed Homotopy Algebras

Here we define OCHA using the same grading and signs conventions of [6] which is the more appropriate for the coalgebra description. For a equivalent description where the grading and signs have a geometrical meaning, see [3]. Let us begin by recalling the definition of L_∞-algebras [11].

Definition 1 (Strong Homotopy Lie algebra). *A strong homotopy Lie algebra (or L_∞-algebra) is a $\mathbb{Z}$-graded vector space V endowed with a collection of degree one graded symmetric n-ary brackets $l_n : V^{\otimes n} \to V$, such that $l_1^2 = 0$ and for $n \geqslant 2$:*

$$\sum_{\sigma \in \Sigma_{k+l=n}} \epsilon(\sigma)\, l_l(l_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}) = 0 \qquad (1)$$

where σ runs over all (k,l)-unshuffles, i.e., permutations $\sigma \in S_n$ such that $\sigma(i) < \sigma(j)$ for $1 \leqslant i < j \leqslant k$ and for $k+1 \leqslant i < j \leqslant k+l$.

Definition 2 (Open-Closed Homotopy Algebra – OCHA). *An OCHA consists of a pair of graded vector spaces $(\mathcal{H}_c, \mathcal{H}_o)$ endowed with two sequences of degree one multilinear operations $\mathfrak{l} = \{l_n : L^{\otimes n} \to L\}_{n \geqslant 1}$ and $\mathfrak{n} = \{n_{p,q} : L^{\otimes p} \otimes A^{\otimes q} \to A\}_{p+q \geqslant 1}$, such that $(L, \mathfrak{l})$ is an L_∞-algebra and the two families satisfy the*

following compatibility condition:

$$\sum_{\sigma\in\Sigma_{p+r=n},\ p\geqslant 2} (-1)^{\epsilon(\sigma)} n_{1+r,m}(l_p(v_{\sigma(1)},\dots,v_{\sigma(p)}),v_{\sigma(p+1)},\dots,v_{\sigma(n)},a_1,\dots,a_m)+$$
$$+\sum_{\substack{\sigma\in\Sigma_{p+r=n}\\ i+j=m-s}} (-1)^{\mu_{p,i}(\sigma)} n_{p,i+1+j}(v_{\sigma(1)},..,v_{\sigma(p)},a_1,..,a_i,$$
$$n_{r,s}(v_{\sigma(p+1)},..,v_{\sigma(n)},a_{i+1},..,a_{i+s}),a_{i+s+1},..,a_m)=0.$$

where $\mu_{p,i}(\sigma)=s+i+si+ms+\epsilon(\sigma)+s(v_{\sigma(1)}+\dots+v_{\sigma(p)}+a_1+\dots+a_i)+$
$+(a_1+\dots+a_i)(v_{\sigma(i+1)})+\dots+v_{\sigma(n)})$.

3 Coderivations

Here we shall briefly recall the notion of coderivation and how the Gerstenhaber bracket can be seen intrinsically as the graded commutators of coderivations on the tensor coalgebra, see [13] for details.

Given a coalgebra (C,Δ,ε), a coderivation of C is a linear map $f: C\to C$ such that $(f\otimes 1+1\otimes f)\Delta=\Delta f$ and $\varepsilon f=0$. In the case of the tensor coalgebra T^cV cogenerated by V, any linear map $f: V^{\otimes k}\to V$ can be lifted to a coderivation $\hat{f}: T^cV\to T^cV$ defined as $\hat{f}(v_1\otimes\dots\otimes v_n)=0$ if $n<k$ and

$$\hat{f}(v_1\otimes\dots\otimes v_n)=\sum_{i=0}^{n-k}(1^{\otimes i}\otimes f\otimes 1^{n-i-k})(v_1\otimes\dots\otimes v_n),\quad \text{for } n\geqslant k. \tag{2}$$

Consequently, any map $f: T^cV\to V$ can be lifted to $\hat{f}: T^cV\to T^cV$ by adding up the liftings of each component of the map f. The lifting as a coderivation defines an isomorphism between the vector spaces $\mathrm{Hom}(T^cV,V)$ and $\mathrm{Coder}(T^cV)$.

Restricting attention to graded coderivations, we have a space with the structure of a graded Lie algebra with bracket given by the graded commutator of compositions: $[\theta,\phi]=\theta\circ\phi-(-1)^{|\theta||\phi|}\phi\circ\theta$. Under the isomorphism provided by the lifting as a coderivation, the above bracket induces a graded Lie algebra structure on $\bigoplus_{n\geqslant 0}\mathrm{Hom}(V^{\otimes n},V)$ (the graded space corresponding to the space of graded coderivations). Stasheff has shown that the bracket induced through the above isomorphism is, up to sign, the Gerstenhaber bracket [13]. The sign can be adjusted using suspension and desuspension operators, see [1, 13].

Observation 1. *It is also possible to define the lifting as a coderivation for the symmetric coalgebra Λ^cV as well as for $\Lambda^cV\otimes T^cU$, see formulas on page 13. However, those formulas holds only in the particular case of the symmetric and tensor coalgebras. For a more general study of coderivations, one can consider coderivations of a $\mathcal{P}$-coalgebra X, where $\mathcal{P}$ is an operad* [12].

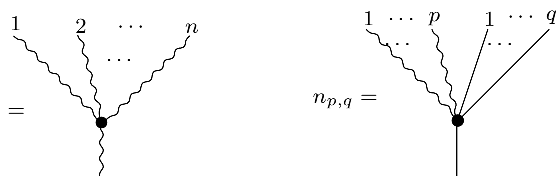
Observation 2. *Since the spaces $\mathcal{H}_c$ and $\mathcal{H}_o$ are graded, there are two ways of grading the space of coderivations. One way is through the* external *degree, i.e.,*

the grading by the number of variables. The second is induced by the grading of $\mathcal{H}_c \otimes \mathcal{H}_o$ *and is called* internal *degree. In this paper we will only use the internal degree.*

3.1 Coderivations on $\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o$

According to its operadic description [3, 6, 7], each OCHA operation $n_{p,q}$ and l_n correspond to a partially planar corolla:

$l_n =$ $\qquad$ $n_{p,q} =$ $\qquad$ (3)



In the language of trees, the OCHA constraint can be stated as follows: "The OCHA operad has no corolla with spatial root and planar leaves". The Axelrod-Singer compactification of the moduli space of points on the closed disc is a manifold with corners whose boundary is labelled by the partially planar trees obtained after grafting corollae of the above type. Other types of partially planar corollae will not appear in that boundary strata. That is the geometrical origin of the OCHA constraint.

In this section we show that any coderivation on $\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o$ satisfies the OCHA constraint, i.e., that the OCHA structure is *intrinsic* to the coalgebra $\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o$. We begin by recalling the definition of the shuffle product.

Definition 3. *For any vector space E, let $a_1 \otimes \cdots \otimes a_n \in E^{\otimes n}$ and $0 \leqslant i \leqslant n$. The* shuffle product *of $a_1 \otimes \cdots \otimes a_i \in E^{\otimes i}$ and $a_{i+1} \otimes \cdots \otimes a_n \in E^{\otimes n-i}$ is:*

$$Sh_{i,n-i}(a_1 \otimes \cdots \otimes a_i | a_{i+1} \otimes \cdots \otimes a_n) := \sum_{\sigma \in \mathfrak{U}_{(i,n-i)}} (-1)^{\epsilon(\sigma)} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \quad (4)$$

where $\mathfrak{U}_{(i,n-i)}$ is the set of all $(i, n-i)$-shuffles, i.e., $\sigma \in \Sigma_n$ with

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(i), \ \sigma^{-1}(i+1) < \cdots < \sigma^{-1}(n).$$

The shuffle product $Sh : T^cE \otimes T^cE \to T^cE$ defines an associative product that is compatible with the deconcatenation coproduct $\Delta : T^cE \to T^cE \otimes T^cE$, so Sh is a coalgebra map with respect to Δ.

Now consider $E = \mathcal{H}_c \oplus \mathcal{H}_o$ and define $\Xi : \Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \to T^c(\mathcal{H}_c \oplus \mathcal{H}_o)$:

$$\Xi((c_1 \wedge \cdots \wedge c_p) \otimes (o_1 \otimes \cdots \otimes o_q)) = Sh(\chi(c_1 \wedge \cdots \wedge c_p)|(o_1 \otimes \cdots \otimes o_q))$$

where $\chi : \Lambda^c\mathcal{H}_c \to T^c\mathcal{H}_c$ is the *symmetrization coalgebra map*:

$$\chi(c_1 \wedge \cdots \wedge c_n) = \sum_{\sigma \in S_n} (-1)^{\epsilon(\sigma)} c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(n)}.$$

We will sometimes refer to $\Xi : \Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \to T^c(\mathcal{H}_c \oplus \mathcal{H}_o)$: as the *symmetrization and shuffling coalgebra map.*

Since we are working over a field of characteristic zero, the symmetrization and shuffling Ξ is an injective coalgebra map. Now let $p_n : \Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \to \bigoplus_{p+q=n} \mathcal{H}_c^{\wedge p} \otimes \mathcal{H}_o^{\otimes q}$ be the canonical projection. We observe that $p_n = \Xi^{-1}\pi_n\Xi$ where $\pi_n : T^c(\mathcal{H}_c \oplus \mathcal{H}_o) \to (\mathcal{H}_c \oplus \mathcal{H}_o)^{\otimes n}$ is the canonical projection. In particular, $p_1 = \pi_1\Xi : \Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \to \mathcal{H}_c \oplus \mathcal{H}_o$. It is not diFfficult to see that $\pi_n = (\pi_1 \otimes \cdots \otimes \pi_1)\Delta_\otimes^{(n-1)}$, Where $\Delta_\otimes$ denotes the deconcatenation coproduct on $T^c(\mathcal{H}_c \oplus \mathcal{H}_o)$, we thus have:

$$\begin{aligned} p_n &= \Xi^{-1}(\pi_1 \otimes \cdots \otimes \pi_1)\Delta_\otimes^{(n-1)}\Xi \\ &= \Xi^{-1}(\pi_1 \otimes \cdots \otimes \pi_1)\Xi^{\otimes n}\Delta^{(n-1)} \\ &= \Xi^{-1}(p_1 \otimes \cdots \otimes p_1)\Delta^{(n-1)} \end{aligned} \tag{5}$$

where Δ is the coproduct on $\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o$ defined by (16) in the appendix.

Given a coderivation $\phi \in \mathrm{Coder}(\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o)$, applying it to (5):

$$\begin{aligned} p_n\phi &= \Xi^{-1}(p_1 \otimes \cdots \otimes p_1)\Delta^{(n-1)}\phi \\ &= \Xi^{-1}\sum(p_1 \otimes \cdots \otimes p_1\phi \otimes \cdots \otimes p_1)\Delta^{(n-1)}, \end{aligned}$$

we conclude that ϕ is determined by its projection $p_1\phi : \Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \to \mathcal{H}_c \oplus \mathcal{H}_o$, $p_1\phi = g \oplus f$. We can write g and f as: $g = \sum g_{p,q}$ and $f = \sum f_{p,q}$, where $g_{p,q} : \mathcal{H}_c^{\wedge p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_c$ and $f_{p,q} : \mathcal{H}_c^{\wedge p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_o$.

Let us show that ϕ is given by the lifting of $g_{p,q}$ and $f_{p,q}$ as coderivations:

$$\begin{aligned} p_n\phi &= \Xi^{-1}\sum(p_1 \otimes \cdots \otimes (f+g) \otimes \cdots \otimes p_1)\Delta^{(n-1)} \\ &= \Xi^{-1}\sum(p_1 \otimes \cdots \otimes (\sum g_{p,q} + \sum f_{p,q}) \otimes \cdots \otimes p_1)\Delta^{(n-1)}, \end{aligned}$$

consequently, $p_n\phi$ is the sum of the two expressions:

$$\sum_{p,q}\Xi^{-1}\sum(p_1 \otimes \cdots \otimes g_{p,q} \otimes \cdots \otimes p_1)\Delta^{(n-1)} \tag{6}$$

$$\sum_{p,q}\Xi^{-1}\sum(p_1 \otimes \cdots \otimes f_{p,q} \otimes \cdots \otimes p_1)\Delta^{(n-1)}. \tag{7}$$

Extending $f_{p,q} \mapsto \hat{f}_{p,q}$ by formula (17), we see that (7) is equal to $p_n \sum_{p,q} \hat{f}_{p,q}$. On the other hand, we lift $g_{p,q} : \mathcal{H}_c^{\wedge p} \otimes \mathcal{H}_o^{\wedge q} \to \mathcal{H}_c$ to $\hat{g}_{p,q} : \Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \to \Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o$, with $\hat{g}_{p,q}$ given by:

$$\hat{g}_{p,q}((u_1 \wedge \cdots \wedge u_n) \otimes (v_1 \otimes \cdots \otimes v_m)) = 0, \qquad \text{if } n < p \text{ or } m < q,$$

and

$$\begin{aligned} &\hat{g}_{p,q}((u_1 \wedge \cdots \wedge u_n) \otimes (v_1 \otimes \cdots \otimes v_m)) = \\ &\sum_{\sigma \in \mathfrak{S}_{p,n-p}} \pm(g_{p,q}(u_{\sigma(1)}, \ldots, u_{\sigma(p)}, v_{i+1}, \ldots, v_{i+q}) \wedge u_{\sigma(p+1)} \wedge \cdots \wedge u_{\sigma(n)}) \otimes \\ &\qquad \otimes (v_1 \otimes \cdots \otimes \widehat{v_{i+1}} \otimes \cdots \otimes \widehat{v_{i+q}} \otimes \cdots \otimes v_m) \end{aligned} \tag{8}$$

where $\pm = (-1)^{(\epsilon(\sigma) + (u_{\sigma(p+1)} + \cdots + u_{\sigma(n)} + v_1 + \cdots + v_i)(v_{i+1} + \cdots + v_{i+q}))}$ and $\hat{v}$ means that v is omitted in the expression.

It follows that $p_n \sum_{p,q} \hat{g}_{p,q}$ is equal to expression (6), and the coderivation can thus be written as $\phi = \sum \hat{g}_{p,q} + \sum \hat{f}_{p,q}$. Observe that (8) reduces to (18) in the appendix when $q = 0$.

We say that a coderivation ϕ satisfies the *OCHA constraint* if

$$\phi = \sum \hat{g}_p + \sum \hat{f}_{p,q} \tag{9}$$

for $g_p : \mathcal{H}_c^{\wedge p} \to \mathcal{H}_c$ and $f_{p,q} : \mathcal{H}_c^{\wedge p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_o$.

Proposition 1. *Every coderivation $\phi \in \mathrm{Coder}(\Lambda^c \mathcal{H}_c \otimes T^c \mathcal{H}_o)$ satisfies the OCHA constraint.*

Proof. We have already seen that formula (5) implies that any coderivation $\phi \in \mathrm{Coder}(\Lambda^c \mathcal{H}_c \otimes T^c \mathcal{H}_o)$ is uniquely written as $\phi = \sum \hat{g}_{p,q} + \sum \hat{f}_{p,q}$ where $\hat{f}_{p,q}$ was defined in formula (17) and $\hat{g}_{p,q}$ in formula (8). We know that $\hat{f}_{p,q}$ is a coderivation for any p, q and that $\hat{g}_{p,q}$ is a coderivation if $q = 0$. Thus $\phi - (\sum \hat{f}_{p,q} + \sum \hat{g}_{p,0}) = \sum\limits_{(p,q) \geqslant (0,1)} \hat{g}_{p,q}$ is a coderivation:

$$\sum_{(p,q) \geqslant (0,1)} (\hat{g}_{p,q} \otimes 1 + 1 \otimes \hat{g}_{p,q}) \bar{\Delta} = \sum_{(p,q) \geqslant (0,1)} \bar{\Delta} \hat{g}_{p,q} \tag{10}$$

where $\bar{\Delta}$ denotes the reduced comultiplication: $\Delta x = x \otimes 1 + \bar{\Delta} x + 1 \otimes x$.

We will prove that $g_{p,q} \equiv 0$ for any $p \geqslant 0, q \geqslant 1$. To simplify the exposition we will use the notation: $u_{[p]} = u_1 \wedge \cdots \wedge u_p$. From (8), we have

$$\hat{g}_{p,q}(u_{[p]} \otimes (v_1 \otimes \cdots \otimes v_{q+1})) = g_{p,q}(u_{[p]}; v_1 \otimes \cdots \otimes v_q) \otimes v_{q+1} \pm g_{p,q}(u_{[p]}; v_2 \otimes \cdots \otimes v_{q+1}) \otimes v_1$$

where $\pm = (-1)^{|v_1|(|v_2| + \cdots + |v_{q+1}|)}$.

Applying $u_{[p]} \otimes (v_1 \otimes \cdots \otimes v_q \otimes v_{q+1})$ to both sides of equation (10) and projecting the result onto $(\mathcal{H}_c \otimes \mathcal{H}_o) \oplus (\mathcal{H}_o \otimes \mathcal{H}_c)$ (viewed as a subspace of $(\Lambda^c \mathcal{H}_c \otimes T^c \mathcal{H}_o) \otimes (\Lambda^c \mathcal{H}_c \otimes T^c \mathcal{H}_o)$) we have:

$$\begin{aligned} g_{p,q}(u_{[p]}; v_1, \ldots, v_q) \otimes v_{q+1} \pm v_1 \otimes g_{p,q}(u_{[p]}; v_2, \ldots, v_{q+1}) = \\ = g_{p,q}(u_{[p]}; v_1, \ldots, v_q) \otimes v_{q+1} \pm g_{p,q}(u_{[p]}; v_2, \ldots, v_{q+1}) \otimes v_1 \pm \\ \pm v_{q+1} \otimes g_{p,q}(u_{[p]}; v_1, \ldots, v_q) \pm v_1 \otimes g_{p,q}(u_{[p]}; v_2, \ldots, v_{q+1}). \end{aligned}$$

where $\pm$ is, in order of appearance: $(-1)^{|v_1|(|u_{[p]}|+1)}$; $(-1)^{|v_{q+1}|(|u_{[p]}| + |v_1| + \cdots + |v_q| + 1)}$; $(-1)^{|v_1|(|v_2| + \cdots + |v_{q+1}|)}$; $(-1)^{|v_1|(|u_{[p]}|+1|)}$.

It follows that:

$$g_{p,q}(u_{[p]}; v_2, \ldots, v_{q+1}) \otimes v_1 \pm v_{q+1} \otimes g_{p,q}(u_{[p]}; v_1, \ldots, v_q) = 0,$$

one summand belongs to $(\mathcal{H}_c \otimes \mathcal{H}_o)$ while the other one belongs to $(\mathcal{H}_o \otimes \mathcal{H}_c)$, hence both are zero. Assuming $v_{q+1} \neq 0$, we have $g_{p,q}(u_{[p]}; v_1, \ldots, v_q) = 0$. $\square$

Theorem 1. *An OCHA structure on the pair $(\mathcal{H}_c, \mathcal{H}_o)$ is equivalent to a degree one coderivation $D \in \mathrm{Coder}^1(\Lambda^c \mathcal{H}_c \otimes T^c \mathcal{H}_o)$ such that $D^2 = 0$.*

3.2 OCHA-morphisms

Consider two families of degree zero linear maps $f_k : \Lambda^k \mathcal{H}_c \to \mathcal{H}'_c$, for $k \geqslant 1$ and $f_{p,q} : \Lambda^p \mathcal{H}_c \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}'_o$ for $p+q \geqslant 1$. Given two OCHAs $(\mathcal{H}_c \oplus \mathcal{H}_o, D)$ and $(\mathcal{H}'_c \oplus \mathcal{H}'_o, D')$, according Kajiura and Stasheff we say that the maps $\{f_k\}_{k \geqslant 1}$ and $\{f_{p,q}\}_{p+q \geqslant 1}$ define an OCHA-morphism when they commute with the OCHA structures after lifted as a coalgebra morphism. More precisely:

$$\mathfrak{f} \circ D = D' \circ \mathfrak{f}$$

where $\mathfrak{f} : \Lambda^c \mathcal{H}_c \otimes T^c \mathcal{H}_o \to \Lambda^c \mathcal{H}'_c \otimes T^c \mathcal{H}'_o$ is the coalgebra morphism obtained by lifting the degree zero linear maps $\{f_k\}_{k \geqslant 1}$ and $\{f_{p,q}\}_{p+q \geqslant 1}$. Explicit formulas for OCHA-morphisms are available in [6]. In the particular case of linear OCHA-morphisms, explicit formulas are provided below.

We say that an OCHA-morphism $\mathfrak{f}$ is *linear* when it is obtained by lifting maps $g : \mathcal{H}_c \to \mathcal{H}'_c$, $f_{0,1} : \mathcal{H}_o \to \mathcal{H}'_o$ and $f_{1,0} : \mathcal{H}_c \to \mathcal{H}'_o$. Denoting the OCHA structures by $D = \mathfrak{l} + \mathfrak{n}$ and $D' = \mathfrak{l}' + \mathfrak{n}'$, equation $\mathfrak{f} \circ D = D' \circ \mathfrak{f}$ can be written as $g \circ l_n = l'_n \circ g^{\otimes n}$ for the L_∞-structure maps and, for the remaining OCHA-structure maps as:

$$f_{0,1}(\eta_{n,0}(c_1, \ldots, c_n)) + f_{1,0}(l_n(c_1, \ldots, c_n)) = \\ = \sum_{p+q=n} \frac{1}{p!}\ \eta'_{p,q}(g^{\otimes p} \otimes f_{1,0}^{\otimes q}) \chi(c_1, \ldots, c_n) \quad (11)$$

$$f_{0,1}(\eta_{p,q}(c_1, \ldots, c_p, o_1, \ldots, o_q)) = \\ = \sum_{0 \leqslant m \leqslant p} \frac{1}{m!}\ \eta'_{m,n}(g^{\otimes m} \otimes Sh_{p-m,q}(f_{1,0}^{\otimes (p-m)} \otimes f_{0,1}^{\otimes q}))(\chi(c_1, .., c_p), o_1, .., o_q). \quad (12)$$

The commutator χ and the shuffle product Sh were defined in section 3.1.

4 Commutators and shuffles of A_∞-extensions

Given an associative algebra with product $a \cdot b$, one can obtain a Lie algebra through the commutator: $[a, b] = a \cdot b - b \cdot a$. This fundamental fact is also true in the context of strongly homotopy Lie algebras. In fact, Lada and Markl [10] have used the symmetrization coalgebra map to relate A_∞ and L_∞ algebras and define the universal enveloping A_∞-algebra of an L_∞-algebra.

In this section we will show that analogous relations hold in the context of OCHA. In other words, we show that an OCHA can be obtained from commutators and shuffles of OCHA constrained A_∞ structures on $\mathcal{H}_c \oplus \mathcal{H}_o$ or, equivalently, A_∞-extensions of $\mathcal{H}_c$ by $\mathcal{H}_o$. The universal enveloping A_∞-algebra of an OCHA is then defined as an A_∞-extension satisfying a natural universal property.

4.1 Constraints on A_∞-algebras as A_∞-extensions

In this section we show that the OCHA constraint on an A_∞-algebra can be understood as an A_∞-extension.

Definition 4. *Let $(E, \mathcal{D})$ be an A_∞-algebra which split in the category of vector spaces as: $E = A \oplus B$. We say that the A_∞-algebra E satisfy the OCHA constraint with respect to the splitting $E = A \oplus B$ if $\mathcal{D}$ has only components of the form: $\mathcal{D}^B_q : B^{\otimes q} \to B$ and $\mathcal{D}^A_{p,q} : A^{\otimes p} \otimes B^{\otimes q} \to A$.*

In this paper, A_∞-extensions are defined using *linear* A_∞-morphisms. A linear A_∞-morphism between two A_∞-algebras $(A, M = \{m_k\})$ and $(B, M' = \{m'_k\})$ is a degree zero linear map $f : A \to B$ such that:

$$f \circ m_k = m'_k \circ (f \otimes \cdots \otimes f) \qquad \forall k \geqslant 1.$$

In general, an A_∞-morphism is given by a degree zero coalgebra morphism $\varphi : T^c(A) \to T^c(B)$ such that $M' \circ \varphi = \varphi \circ M$, viewing M and M' as coderivation differentials defining the A_∞-structures.

Definition 5 (A_∞-ideal)**.** *Let A be an A_∞-algebra. An A_∞-ideal of A is a subespace $I \subseteq A$ such that, for any $k \geqslant 1$, $m_k(x_1, \ldots, x_k) \in A$ whenever $x_i \in I$ for some $1 \leqslant i \leqslant k$.*

Notice that an A_∞-ideal is in particular a subcomplex of A and, if $f : A \to B$ is a linear A_∞-morphism, then $\mathrm{Ker}(f) \subset A$ is an A_∞-ideal of A.

Definition 6. *Let A and B be A_∞-algebras. We say that an A_∞-algebra E is an A_∞-extension of B by A if there exists an exact sequence*

$$0 \to A \to E \to B \to 0$$

where each map is a linear A_∞-morphism.

If E is an extension of B by A, then $E = A \oplus B$ as vector spaces and, since A is an A_∞-ideal in E, we see that the A_∞-algebra E satisfies the OCHA constraint with respect to the splitting $E = A \oplus B$. Now that the appropriate concepts are given, we can just apply a well known argument [9, 10] to prove the following theorem.

Theorem 2. *Let A and B be two A_∞-algebras. If $(E, \mathcal{D})$ is an A_∞-extension of B by A, then $\mathcal{D} \circ \Xi$ defines an OCHA structure on the pair (B, A). The L_∞ structure on B is the Lada-Markl symmetrization of the A_∞ structure on B.*

Proof. Since $\mathcal{D} = \{\mathcal{D}_k\}$ is OCHA constrained, the composition $\mathcal{D} \circ \Xi = \{\mathcal{D}_k \circ \Xi\}$ gives two sequence of maps: $l_k = \ell_k \circ \Xi : \Lambda^k \mathcal{H}_c \to \mathcal{H}_c$ and $n_{p,q} = n_k \circ \Xi : \mathcal{H}_c^{\wedge p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_o$ with $p+q = k$. Equations (17) and (18) tells us how to lift these maps as coderivations $\mathfrak{l}_k \in \mathrm{Coder}(\Lambda^c \mathcal{H}_c \otimes T^c \mathcal{H}_o)$ and $\mathfrak{n}_{p,q} \in \mathrm{Coder}(\Lambda^c \mathcal{H}_c \otimes T^c \mathcal{H}_o)$. We denote by $\widehat{\mathcal{D} \circ \Xi}$ the coderivation $\mathfrak{l} + \mathfrak{n} = \sum \mathfrak{l}_k + \sum \mathfrak{n}_{p,q} \in \mathrm{Coder}(\Lambda^c \mathcal{H}_c \otimes T^c \mathcal{H}_o)$.

Let us now consider the following diagram:

$$\begin{tikzcd}
\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \arrow[r, "\Xi"] & T^c(\mathcal{H}_c \oplus \mathcal{H}_o) & \\
\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \arrow[r, "\Xi"] \arrow[u, "\widehat{\mathcal{D}\circ\Xi}"] & T^c(\mathcal{H}_c \oplus \mathcal{H}_o) \arrow[u, "\hat{\mathcal{D}}"'] \arrow[dr, "\pi_c\oplus\pi_o"] & \\
\Lambda^c\mathcal{H}_c \otimes T^c\mathcal{H}_o \arrow[r, "\Xi"] \arrow[u, "\widehat{\mathcal{D}\circ\Xi}"] & T^c(\mathcal{H}_c \oplus \mathcal{H}_o) \arrow[r, "\mathcal{D}"] \arrow[u, "\hat{\mathcal{D}}"'] & \mathcal{H}_c \oplus \mathcal{H}_o,
\end{tikzcd} \tag{13}$$

Since $\mathcal{D}$ is OCHA constrained, it is not dificult to check that $\Xi \circ \widehat{\mathcal{D}\circ\Xi} = \widehat{\mathcal{D}} \circ \Xi$, so the diagram is commuatative. Since $\mathcal{D} = \{\mathcal{D}_k\}$ defines an A_∞-algebra structure, we have $\widehat{\mathcal{D}}^2 = 0$. Since Ξ is injective, from the above diagram we have $(\widehat{\mathcal{D}\circ\Xi})^2 = (\mathfrak{l}+\mathfrak{n})^2 = 0$. $\square$

Observation 3. *If E is an A_∞-extension of B by A, we will denote the OCHA obtained through commutators and shuffles by $(E)_{OC}$.*

4.2 Universal Enveloping A_∞-algebra of an OCHA

Let us now construct the universal enveloping A_∞ algebra of an OCHA. Given an OCHA structure $\mathfrak{l}+\mathfrak{n} = \sum_{n\geqslant 1} l_n + \sum_{p+q\geqslant 1} n_{p,q}$ on a pair $(\mathcal{H}_c, \mathcal{H}_o)$, let $\mathcal{F}_\infty(\mathcal{H}_c \oplus \mathcal{H}_o)$ be the free A_∞ algebra generated by $\mathcal{H}_c \oplus \mathcal{H}_o$ with A_∞ structure maps denoted by $\mu_n : (\mathcal{H}_c \oplus \mathcal{H}_o)^{\otimes n} \to \mathcal{H}_c \oplus \mathcal{H}_o$.

We define the universal enveloping A_∞-algebra $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)$ as the quotient of $\mathcal{F}_\infty(\mathcal{H}_c \oplus \mathcal{H}_o)$ by the A_∞-ideal I generated by the relations:

$$\sum_{\sigma\in S_p} (-1)^{\epsilon(\sigma)} \mu_{p+q}(Sh(c_{\sigma(1)}, .., c_{\sigma(p)}|o_1, .., o_q)) = n_{p,q}(c_1, .., c_p, o_1, .., o_q) \tag{14}$$

$$\sum_{\sigma\in S_p} (-1)^{\epsilon(\sigma)} \mu_p(c_{\sigma(1)}, \dots, c_{\sigma(p)}) = l_p(c_1, \dots, c_p) + n_{p,0}(c_1, \dots, c_p), \tag{15}$$

for $p \geqslant 0$, $q \geqslant 1$ in (14) and $p \geqslant 1$ in (15), where $c_i \in \mathcal{H}_c$ and $o_j \in \mathcal{H}_o$.

In case $\mathcal{H}_o = 0$, the OCHA structure reduces to an L_∞-algebra structure on $\mathcal{H}_c$ and the above construction reduces to the universal enveloping A_∞-algebra $\mathcal{U}_\infty(\mathcal{H}_c)$ of an L_∞-algebra introduced by Lada and Markl in [10]. In general, we have the following result relating the two constructions.

Theorem 3. *The universal enveloping A_∞-algebra $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)$ of an OCHA $(\mathcal{H}_c, \mathcal{H}_o, \mathcal{D})$ is an A_∞-extension of $\mathcal{U}_\infty(\mathcal{H}_c)$ by $\langle\mathcal{H}_o\rangle$:*

$$0 \to \langle\mathcal{H}_o\rangle \to \mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o) \to \mathcal{U}_\infty(\mathcal{H}_c) \to 0$$

where $\langle\mathcal{H}_o\rangle$ is the A_∞-ideal generated by $\mathcal{H}_o$.

Proof. We just need to show that $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)/\langle\mathcal{H}_o\rangle$ is isomorphic to $\mathcal{U}_\infty(\mathcal{H}_c)$, i.e., that $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)/\langle\mathcal{H}_o\rangle$ satisfies the universal property defining $\mathcal{U}_\infty(\mathcal{H}_c)$. Let A be an A_∞-algebra and let $f : \mathcal{H}_c \to A$ be any linear map inducing a L_∞-morphism from $\mathcal{H}_c$ to $(A)_L$ (the L_∞-algebra defined by commutators of A). Since $\mathcal{F}_\infty(\mathcal{H}_c \oplus \mathcal{H}_o)$ is free, there is a unique A_∞-morphism from $\mathcal{F}_\infty(\mathcal{H}_c \oplus \mathcal{H}_o)$ to A extending f and vanishing on $\langle\mathcal{H}_o\rangle$. To see that it also vanishes on the ideal I of relations defining $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)$, we just need to note that it satisfies relations (14) because it vanishes on $\langle\mathcal{H}_o\rangle$ and relations (15) since $f : \mathcal{H}_c \to (A)_L$ is an L_∞-morphism. $\square$

The universal property characterizing $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)$ is described as follows. Let $(\mathcal{H}_c, \mathcal{H}_o, \mathcal{D})$ be an OCHA and let A and B be A_∞-algebras. For any A_∞-extension E of B by A and any linear map $\mathcal{H}_c \oplus \mathcal{H}_o \xrightarrow{f} E$ such that $\mathcal{H}_c \oplus \mathcal{H}_o \xrightarrow{f} (E)_{OC}$ is a linear OCHA-morphism, there exists a unique morphism of A_∞-extensions $\varphi : \mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o) \to E$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & \mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o) \arrow[d, "\varphi"] \\
\mathcal{H}_c \oplus \mathcal{H}_o \arrow[ur, "\iota"] \arrow[r, "f"] & E
\end{tikzcd}$$

where $\iota : \mathcal{H}_c \oplus \mathcal{H}_o \to \mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)$ is the inclusion. Since $\mathcal{F}_\infty(\mathcal{H}_c \oplus \mathcal{H}_o)$ is free, there is a unique A_∞-morphism $\varphi : \mathcal{F}_\infty(\mathcal{H}_c \oplus \mathcal{H}_o) \to E$ extending f. It vanishes on the ideal of OCHA relations (14) and (15) because $f : \mathcal{H}_c \oplus \mathcal{H}_o \to E_{OC}$ is a linear OCHA morphism. Hence φ is well defined on $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)$. This proves that there is a unique A_∞-morphism $\varphi : \mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o) \to E$ extending f.

To see that φ is a morphism of A_∞-extensions, we just need to observe that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \langle\mathcal{H}_o\rangle \arrow[r] \arrow[d] & \mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o) \arrow[r] \arrow[d, "\varphi"'] & \mathcal{U}_\infty(\mathcal{H}_c) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & E \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

In fact, $\varphi : \mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o) \to E$ is an extension of an OCHA morphism $f : \mathcal{H}_c \oplus \mathcal{H}_o \to (E)_{OC}$. So, it respects the OCHA constraint taking the ideal $\langle\mathcal{H}_o\rangle$ into the ideal A and is thus well defined on the quotient $\mathcal{U}_\infty(\mathcal{H}_c, \mathcal{H}_o)/\langle\mathcal{H}_o\rangle \to E/A$.

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Appendix: Lifting as a coderivation

Here we provide some formulas for the lifting as a coderivation in the case of the coalgebras $\Lambda^c U$ and $\Lambda^c U \otimes T^c \mathcal{H}_o$. U and $\mathcal{H}_o$ are $\mathbb{Z}$-graded vector spaces. The coproduct Δ on $\Lambda^c U \otimes T^c \mathcal{H}_o$ is given explicitly by:

$$\begin{aligned}
&\Delta((u_1 \wedge \cdots \wedge u_m) \otimes (v_1 \otimes \cdots \otimes v_n)) = \\
&= \sum_{\substack{0 \leqslant p \leqslant m \\ 0 \leqslant q \leqslant n}} \sum_{\sigma \in \mathfrak{S}_{p,m-p}} (-1)^{\epsilon(\sigma)} (-1)^{\eta(p,q)} ((u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(p)}) \otimes (v_1 \otimes \cdots \otimes v_q)) \otimes \\
&\qquad \otimes ((u_{\sigma(p+1)} \wedge \cdots \wedge u_{\sigma(m)}) \otimes (v_{q+1} \otimes \cdots \otimes v_n)),
\end{aligned} \tag{16}$$

where : $\eta(p,q) = (u_{\sigma(p+1)} + \cdots + u_{\sigma(m)})(v_1 + \cdots + v_q)$.

Given a map $f : U^{\wedge p} \otimes \mathcal{H}_o{}^{\otimes q} \to \mathcal{H}_o$, we may lift it as a coderivation in the following way: for $r \geqslant p, s \geqslant q$ we define:

$$\begin{aligned}
&\hat{f}((u_1 \wedge \cdots \wedge u_r) \otimes (v_1 \otimes \cdots \otimes v_s)) = \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_{r-p,p} \\ 0 \leqslant j \leqslant s-q}} (-1)^{\mu_{r-p,j}(\sigma)} (u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(r-p)}) \otimes (v_1 \otimes \cdots \otimes v_j \otimes \\
&\qquad \otimes f(u_{\sigma(r-p+1)}, \ldots, u_{\sigma(r)}, v_{j+1}, \ldots, v_{j+q}) \otimes \cdots \otimes v_s)
\end{aligned} \tag{17}$$

where: $\mu_{p,q}(\sigma) = \epsilon(\sigma) + (u_{\sigma(1)} + \cdots + u_{\sigma(p)}) + (v_1 + \cdots + v_q) +$
$$+ (v_1 + \cdots + v_q)(u_{\sigma(q+1)} + \cdots + u_{\sigma(n)}).$$

It is not difficult to check that $\hat{f}$ is a coderivation.

Recall that a map $g : U^{\wedge p} \to U$ may be lifted as a coderivation $\hat{g} : \Lambda^c U \to \Lambda^c U$ so that: $\hat{g}(u_1 \wedge \cdots \wedge u_n) = 0$ for $n < p$ and for $n \geqslant p$ is defined by:

$$\hat{g}(u_1 \wedge \cdots \wedge u_n) = \sum_{\sigma \in \mathfrak{S}_{p,n-p}} (-1)^{\epsilon(\sigma)} g(u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(p)}) \wedge u_{\sigma(p+1)} \wedge \cdots \wedge u_{\sigma(n)}.$$

g can be lifted as a coderivation of $\Lambda^c U \otimes T^c \mathcal{H}_o$ by tensoring the above map with the identity of $T^c \mathcal{H}_o$. We thus have $\hat{g} : \Lambda^c U \otimes T^c \mathcal{H}_o \to \Lambda^c U \otimes T^c \mathcal{H}_o$

$$\begin{aligned}
&\hat{g}((u_1 \wedge \cdots \wedge u_n) \otimes (v_1 \otimes \cdots \otimes v_p)) = \\
&\sum_{\sigma \in \mathfrak{S}_{p,n-p}} (-1)^{\epsilon(\sigma)} (g(u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(p)}) \wedge u_{\sigma(p+1)} \wedge \cdots \wedge u_{\sigma(n)}) \otimes (v_1 \otimes \cdots \otimes v_p).
\end{aligned} \tag{18}$$

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